

VAN DER PAUW METHOD APPLICATION FOR ANISOTROPICALLY CONDUCTIVE MEDIUM

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The paper deals with the ways of finding an electrical conductivity tensor of a plane and anisotropically conductive sample. Application of the van der Pauw method to investigate the conductivity of anisotropically conductive media makes the basis of research. Several models of distribution of the electric field potential are presented, their merits and demerits are discussed, and the necessary physical measurements are indicated. On the basis of these models, the respective calculation expressions of the specific conductivity tensor are derived and algorithms for their realization and error calculation are developed.

Keywords: anisotropic media, electrical conductivity, current flow, measurements

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1. Introduction

In order to measure the specific electrical conductivity of an isotropic substance one can apply the van der Pauw method [1, 2]. This well-known method in the Hall effect theory stands out by its simplicity of measurements and mathematical calculations, and what is most important, it applies to plane sample of any shape. To realize it, we use the plate of uniform thickness d with four point contacts arbitrary located on its contour. When the electric current is flowing via two adjacent contacts, we measure the difference of potentials appearing between the other two contacts (see Fig. 1). Afterwards we repeat the experiment by skipping to the next current contact. Van der Pauw has proved [1] that specific conductivity is the solution to the equation

$$\exp\left(-\frac{\pi s d |\Delta\varphi_{34}|}{I_{12}}\right) + \exp\left(-\frac{\pi s d |\Delta\varphi_{23}|}{I_{14}}\right) = 1 \quad (1)$$

and showed the way of its solution based on the values of some empiric function [1, 2]. It is worth mentioning that equation (1) can be solved in a simpler way, i. e., by applying Newton's method, which always converges in case the initial value $s^{(0)} = 0$. The rate of convergence is rather high, therefore a few arithmetic operations are enough.

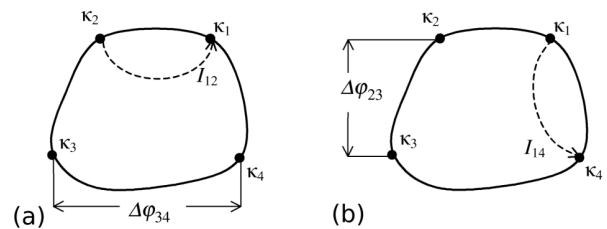


Fig. 1. Measurements of potential differences and current intensities by the van der Pauw method.

In case the conductivity of the substance considered is anisotropic, it is defined by the specific conductivity tensor [3]:

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \quad \det\sigma = \sigma_{11}\sigma_{22} - (\sigma_{12})^2 > 0. \quad (2)$$

Note that in this case, using van der Pauw method, it is possible to find one invariant of the tensor (2). Indeed, let the plane G , corresponding to the shape of a sample with anisotropic conductivity, be located on the coordinate system Oxy . Then, after the affine transformation

$$\tilde{x} = x\sigma_{22} - y\sigma_{12}, \quad \tilde{y} = y\sqrt{\det\sigma},$$

we obtain another domain \tilde{G} in which the conductivity is isotropic, therefore it is defined by a single number s . It can be proven that this number does not depend on the orientation of the principal axes of tensor σ and

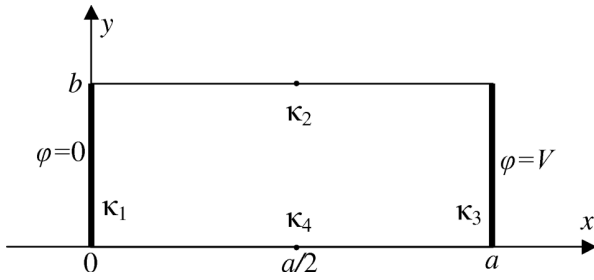


Fig. 2. Arrangement of contacts in a rectangular-shaped sample.

is equal to $\sqrt{\det\sigma}$ (note that a particular case of this statement, where the principal directions of σ are parallel to the axes of the coordinate system, are considered in [4]). Thus, based on this statement and on the fact that the solution to Eq. (1) does not depend on the geometrical shape of a sample and on the contacts location, we find that, after van der Pauw measurements in the sample with anisotropic conductivity, the solution to Eq. (1) is equal to

$$s = \sqrt{\det\sigma}.$$

2. Establishment of tensor components based on a linear model

In order to calculate the components σ_{11} , σ_{12} , σ_{22} of specific conductivity tensor (2), it is necessary to take additional measurements, because using the above mentioned method one can find only the tensor's determinant $\det\sigma$. To solve this problem, we employ a rectangular-shaped sample with four electrodes: two of these are located on the opposite sides of the rectangle and the other two at the central points of the remaining sides (Fig. 2).

When the electric current flows through the contacts κ_1 and κ_3 , we measure the difference of potentials between the contacts κ_2 and κ_4 . If the conductivity of sample is isotropic, then the potential inside the sample is distributed linearly, $\varphi(x, y) = Vx/a$, therefore there appears no difference in potentials of the contacts κ_2 and κ_4 :

$$\Delta\varphi = \varphi(a/2, 0) - \varphi(a/2, b) = 0.$$

On the contrary, anisotropic conductivity is defined by the non-zero difference of potentials. It is expressed in a more complicated way by using the Schwartz–Christoffel integrals (see a more detailed consideration below). However, if the ratio of the sample dimensions is $b/a \ll 1$, then the distribution of potential in the central part of the sample ($x \approx a/2$) is close to the linear

one since nonlinearity is displayed only in the vicinity of current contacts κ_1 and κ_3 (the boundary effect). Therefore we make an assumption that in the zone of the contacts κ_2 and κ_4 the potential is defined by a linear function:

$$\varphi(x, y) = \frac{V}{a} \left[x - \left(y - \frac{b}{2} \right) \frac{\sigma_{12}}{\sigma_{22}} \right], \quad x \approx \frac{a}{2}. \quad (3)$$

This function

- satisfies the current continuity equation

$$\operatorname{div}\sigma \operatorname{grad}\varphi = 0,$$

- satisfies the condition of current nonflow over the sides $y = 0$ and $y = b$

$$(\sigma \operatorname{grad}\varphi)_y \Big|_{y=0} = 0, \quad (\sigma \operatorname{grad}\varphi)_y \Big|_{y=b} = 0,$$

- retains the symmetry of the potential

$$\varphi\left(\frac{a}{2} + x, \frac{b}{2} + y\right) = V - \varphi\left(\frac{a}{2} - x, \frac{b}{2} - y\right).$$

Based on (3) we obtain expressions of the intensity I_{13} of current flowing through the contacts κ_1 and κ_3 , and of the difference of potentials:

$$I_{13} = d \int_0^b (\sigma \operatorname{grad}\varphi)_x dy = Vd \frac{b \det\sigma}{a \sigma_{22}},$$

$$\Delta\varphi = \varphi\left(\frac{a}{2}, 0\right) - \varphi\left(\frac{a}{2}, b\right) = V \frac{b \sigma_{12}}{a \sigma_{22}},$$

as well as the expressions of components in search:

$$\begin{cases} \sigma_{22} = Vd \frac{b \det\sigma}{a I_{13}}, \\ \sigma_{12} = d \Delta\varphi_{24} \frac{\det\sigma}{I_{13}}, \\ \sigma_{11} = \frac{\det\sigma + \sigma_{12}^2}{\sigma_{22}}. \end{cases} \quad (4)$$

Since the determinant of the tensor can be calculated by the van der Pauw method, after measuring the current intensity I_0 and the difference of potentials $\Delta\varphi$, by virtue of (4) one can calculate all the components of the conductivity tensor (2).

To estimate the accuracy of the method, we consider the values of relative error

$$\delta_\sigma = \sqrt{\frac{\sum (s_{ij} - \sigma_{ij})^2}{\sum (s_{ij}^2)}} \cdot 100\%. \quad (5)$$

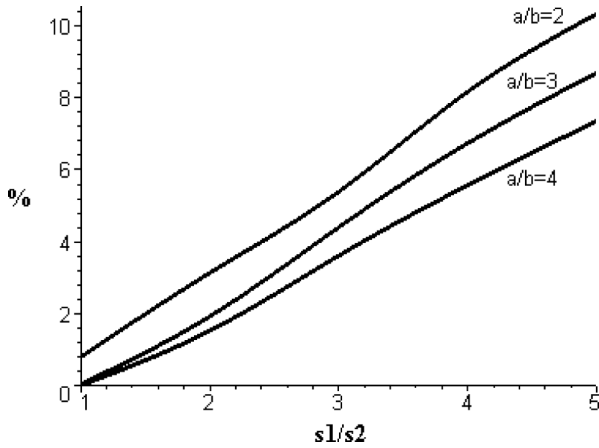


Fig. 3. Dependence of relative errors on the ratio a/b between lengths of sample sides of the conductivity tensor and the ratio s_1/s_2 of principal conductivities (linear model).

Here σ_{ij} are exact values of components of the conductivity tensor and s_{ij} are those calculated by formulas (4). Figure 3 illustrates the variation of maximal errors δ_σ (with respect to the angle comprised by the sample sides and the principal axes) dependent on the ratio between lengths of the sides and ratio between principal conductivities.

As we can see from Fig. 3, relative errors change almost linearly and little depend on dimensions of the sample, moreover, under low anisotropy ($s_1/s_2 \approx 1$) of the medium considered, they are close to zero. That is why, in view of simplicity of calculations performed, this method is practical in case high precision is not necessary, or if it is known beforehand that the medium is proximal to the isotropic one. In other cases the possible errors do not exceed 10%.

3. Measurement of a specific conductivity tensor values including the boundary effect

Let us consider the other method for finding a tensor that is characterized by a higher accuracy, however, its realization requires a lot of (and more complicated) mathematical operations. Its basis lies in the fact that the intensity of the current flowing in the sample and values of potential are established exactly at the points of the sample contour.

We find the expressions of these quantities by defining the functions

$$f_{\alpha,k} = \tau^{\alpha-1} (1 - \tau)^{-\alpha} (1 - k\tau)^{\alpha-1}$$

and integrals

$$A_{\alpha,k}(t) = \int_0^t f_{\alpha,k} d\tau.$$

If the current is flowing through the contacts κ_1 and κ_3 (Fig. 2), then the potential in the sample side $y = 0$ is distributed according to the law [5]

$$\varphi(x, 0) = V \frac{A_{0.5,k}(t)}{A_{0.5,k}(1)}, \quad (6)$$

$$x = a \frac{A_{\alpha,k}(t)}{A_{\alpha,k}(1)}, \quad (7)$$

here

$$t \in [0, 1], \quad \alpha = \frac{1}{\pi} \arccos \frac{-\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}, \quad (8)$$

and the parameter $k \in (0, 1)$ is found from the uniquely solved equation

$$k : \frac{A_{\alpha,k}(1)}{A_{\alpha,k-1}(1)} = \frac{a}{b} \sqrt{\frac{\sigma_{22}}{\sigma_{11}}}.$$

Besides, the current flowing in the sample is

$$I_{13} = V \sqrt{\det \sigma} \frac{A_{0.5,1-k}(1)}{A_{0.5,k}(1)}. \quad (9)$$

Based on (8) and (9), we obtain now the formulas for calculating the components of the conductivity tensor in quest:

$$\begin{cases} \sigma_{11} = \lambda \frac{a}{b} \frac{A_{\alpha,1-k}}{A_{\alpha,k}}, \\ \sigma_{12} = -\lambda \cos(\alpha\pi), \\ \sigma_{22} = \lambda \frac{a}{b} \frac{A_{\alpha,k}}{A_{\alpha,1-k}}, \end{cases} \quad (10)$$

with $\lambda = \sqrt{\det \sigma} / \sin(\alpha\pi)$.

Consequently, if we know the three quantities $\det \sigma$, k , and α , we can calculate the conductivity tensor by (10). We present an algorithm for calculating these quantities. It turns out that they can be calculated by solving in turn the nonlinear equations:

1. $\det \sigma$ is found by measuring according to the van der Pauw method and after solving Eq. (1) (here $s = \sqrt{\det \sigma}$).
2. The parameter k is calculated from Eq. (9) according to calculated $\det \sigma$ and measured current intensity I_{13} .

3. The search of parameter α splits in two parts. First we estimate the value of the potential at the central point $x = a/2$ and then find the value of variable t by the equation (see (6))

$$\varphi\left(\frac{a}{2}, 0\right) = V \frac{A_{0.5,k}(t)}{A_{0.5,k}(1)}.$$

Next, the parameter $\alpha \in (0, 1)$ is found by solving Eq. (7) at $x = a/2$:

$$\frac{A_{\alpha,k}(t)}{A_{\alpha,k}(1)} = \frac{1}{2}.$$

Note that in order to achieve a higher accuracy in the experiments, it is customary to measure the differences of potentials

$$\Delta\varphi_{23} = \varphi\left(\frac{a}{2}, 0\right) - \varphi\left(\frac{a}{2}, b\right),$$

and then the potential $\varphi(a/2, 0)$ is calculated using the relation $\varphi(a/2, 0) + \varphi(a/2, b) = V$.

It can be proven that all the equations above can be solved in a single-valued way, besides, the localizations of their roots are known. Authors have made up the root module for their solution, i. e., for finding tensor components according to the measurement results of currents I_{12} , I_{14} , I_{13} and the respective differences of potentials $\Delta\varphi_{34}$, $\Delta\varphi_{23}$, $\Delta\varphi_{24}$. It takes several seconds to carry out this procedure by applying modern computers.

To evaluate the accuracy of the method, relative errors (5) are calculated under the different ratios of the main components of the conductivity tensor and between lengths of sample sides. Separate graphs have been made up to compare the obtained errors with the respective ones obtained by applying the linear model (Fig. 4). One can see from these that the nonlinear model is more exact, though for $a/b = 2$ the errors are in fact equal, however, with an increase in sample extension a/b , adequacy of the nonlinear model increases. When using samples with $a/b = 4$, the relative error does not exceed 0.5%, which most frequently corresponds to the accuracy of physical measurements.

Sometimes (mostly due to technological circumstances) we have to use samples with their geometrical shape similar to a square [6], i. e., $a/b \approx 1$. In this case, by applying a linear or nonlinear model, errors grow up to several tens of percent and the results of calculations made are inadequate even approximately to the values of components of the conductivity tensor of the present sample. Taking into consideration this fact, we present

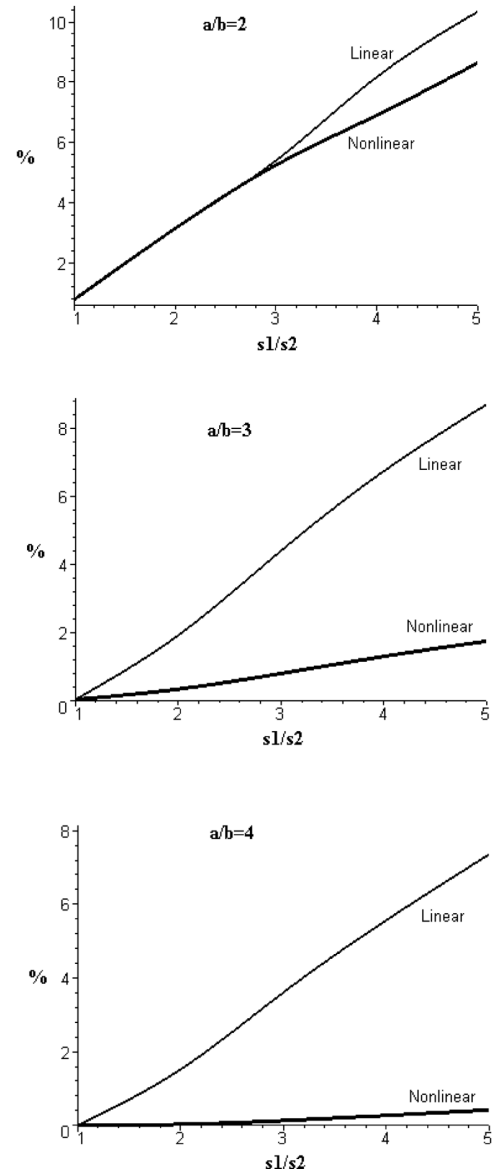


Fig. 4. Distribution of relative errors of a conductivity tensor depending on the ratio a/b of sample dimensions and the ratio s_1/s_2 of the main components of the tensor. For comparison of the methods, the thin lines denote errors calculated by linear model, and the thick lines denote those obtained in the nonlinear model.

below some methods by means of which one can essentially decrease the mentioned error (theoretically up to zero).

4. The specification method

The only reason for emergence of errors in the nonlinear method analyzed (excluding the errors of physical measurements) is that when calculating conductivity $\sqrt{\det\sigma}$, the contacts κ_1 and κ_3 of sample are of non-zero length. It turns out, however, that using samples of a definite geometrical shape (e. g. rectangular-shaped)

it is possible to build a numerical algorithm that after carrying it out for several times allows one to diminish the error considerably.

The basis of algorithm lies in the possibility to obtain exact analytic expressions of the parameters used in the van der Pauw method. At first, let the current I_{12} flow between the contacts κ_1 and κ_2 , and the difference appearing in potentials between the contacts κ_3 and κ_4 is $\Delta\varphi_{34}$. Then the first van der Pauw parameter vdP_1 is

$$\text{vdP}_1 = d \frac{|\Delta\varphi_{34}|}{I_{12}} = \frac{\int_0^{t_2} |F_{\alpha,k}| dt}{\int_{-\infty}^0 |F_{\alpha,k}| dt} \frac{1}{\sqrt{\det\sigma}}, \quad (11)$$

here the function

$$F_{\alpha,k} = \frac{T - t}{|t - t_4| \sqrt{|t(1-t)(1-kt)|}},$$

its parameters t_2, t_4 are found from equations (see (7))

$$t_4 \in (0, 1) : \quad x_4 = \frac{a}{2} = a \frac{\int_0^{t_4} |f_{\alpha,k}| dt}{\int_0^1 f_{\alpha,k} dt},$$

$$t_2 \in \left(\frac{1}{k}, \infty\right) : \quad x_2 = \frac{a}{2} = a \frac{\int_0^{t_2} |f_{\alpha,k}| dt}{\int_0^1 f_{\alpha,k} dt},$$

and number T is found from the condition that the total quantity of current flowing through the contact κ_3 is equal to zero:

$$T : \quad \int_1^{1/k} F_{\alpha,k} dt = 0.$$

The second van der Pauw parameter vdP_2 is calculated replacing α by $1 - \alpha$ in the above formulas:

$$\text{vdP}_2 = d \frac{|\Delta\varphi_{23}|}{I_{14}} = \frac{\int_0^{t_2} |F_{1-\alpha,k}| dt}{\int_{-\infty}^0 |F_{1-\alpha,k}| dt} \frac{1}{\sqrt{\det\sigma}}. \quad (12)$$

Let us describe now the algorithm for decreasing errors. Let $\sigma^{(0)}$ be an approximate value of tensor σ obtained by applying the nonlinear method. Then, based on the obtained formulas (11) and (12), we calculate the parameters $\text{vdP}_1, \text{vdP}_2$ and, having solved Eq. (1),

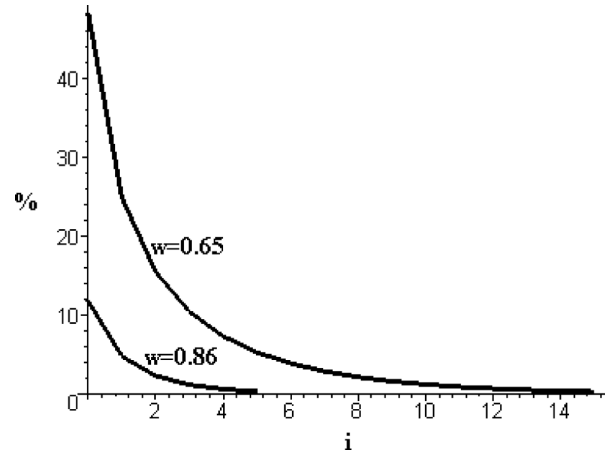


Fig. 5. Dependence of the relative error on the number i of iterations. The upper curve ($w = 0.65$) is obtained when the ratio of sample dimensions $a/b = 1$ and the conductivity tensor $\sigma = \begin{pmatrix} 2.8250 & 0.5646 \\ 0.5646 & 1.1750 \end{pmatrix}$. In this case, the initial error is equal to 48.22%, while from the 13th iteration the errors do not exceed 0.5%. The lower curve ($w = 0.86$) corresponds to $a/b = 2$, $\sigma = \begin{pmatrix} 2.8250 & 0.5646 \\ 0.5646 & 1.1750 \end{pmatrix}$. The initial error equals 11.8%, and from the 5th iteration the errors do not exceed 0.35%.

we obtain another value of conductivity $s = \sqrt{\det\sigma}$. Let us denote it by $s^{(1)}$. After performing all the calculations with it by formulas (10), as described in Sect. 3, we find a new conductivity tensor $\sigma^{(1)}$.

We can repeat this sequence of calculations: if we get the conductivity value at the i th step, then we can find the components of the tensor by formulas (10), by attributing to conductivity the value $\sqrt{\det\sigma}$

$$\sqrt{\det\sigma} = \sqrt{\det\sigma^{(0)}} \frac{s^{(i)}}{s^{(i-1)}}, \quad i = 2, 3, \dots$$

Calculations are repeated as long as the ratio $s^{(i)}/s^{(i-1)}$ becomes stable, i.e., until the difference $|s^{(i)}/s^{(i-1)} - s^{(i+1)}/s^{(i)}|$ becomes small enough.

Figure 5 illustrates the variation of the relative error of the tensor depending on the number i of iterations performed. Note that the rate of decreasing of error, mainly depends on the value of $w = a/b \cdot \sqrt{\sigma_{22}/\sigma_{11}}$, in addition, the greater the w , the less iterations have to be performed.

5. Conclusions

1. This work is meant for extension of the van der Pauw method in calculating the conductivity of an anisotropic substance. It has been proved that one can find an invariant $\sqrt{\det\sigma}$ of conductivity tensor in a sample with anisotropic conductivity, after measuring by the van der Pauw method and

solving Eq. (1). However, to find the complete tensor, it is necessary to form two independent equations in addition. To establish these, other than in the van der Pauw method, we use a rectangular-shaped sample with two contacts of non-zero length and we indicate various ways of forming these equations. To compare their adequacy, the analysis of a relative error of the calculated tensor is applied, dependent on the ratio a/b of sample dimensions and the ratio s_1/s_2 of the principal components of the tensor.

- Using the linear model (i. e., assuming that a rectangular-shaped sample is rather long), we have obtained rather simple formulas for calculating the components of the tensor. The analysis of accuracy made has shown that for $a/b > 2$ the error essentially depends only on the ratio s_1/s_2 , and it amounts up to 10% for $s_1/s_2 = 5$. On the other hand, under low anisotropy ($s_1/s_2 < 2$) the error hardly amounts to 2–3%. Thus, this method stands out by simplicity of calculations, however, its accuracy is not comparatively high.
- When the ratio of sample dimensions is $a/b > 3$ and that of the main conductivities of the substance under consideration is $s_1/s_2 < 5$, it is worth applying a nonlinear model, since in this case the relative error does not exceed 1%. This model relies on the fact that distribution of the potential in a sample is calculated exactly. Then, to find a tensor, computers are applied, and calculations are performed rapidly, in 5–10 seconds.
- As an extension of the method mentioned above, the specification method of iterations can be as-

sumed, which applies to samples of arbitrary dimensions and the substance of which is of any anisotropy, because it provides for a decrease in the error of the conductivity tensor up to zero. To realize it, it is indispensable to use a computer, and the time of calculations directly depends on the number of iterations. For instance, when $2 < a/b < 4$ and $s_1/s_2 < 5$, the accuracy of 0.1% is achieved after 1–3 iterations, while the duration of calculation is about 30 seconds. In a more complicated case where a sample is square-shaped ($a/b = 1$) and $s_1/s_2 = 7$, after 13 iterations the error diminishes from 48% up to 0.5%, while the duration of calculation is 3 minutes (processor Pentium 4, 3 GHz).

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VAN DER PAUW METODO TAIKYMAS ANIZOTROPIŠKAI LAIDŽIAI TERPEI

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Santrauka

Darbas skirtas anizotropiškai laidžios terpės savitojo elektrinio laidumo tenzoriui rasti. Skirtingai nuo klasikinių būdų, kai naudojami keli bandiniai, parodoma, kad pagaminus vieną stačiakampio formos bandinį ir taikant van der Pauw metodą, ieškomą tenzorių galima apskaičiuoti pakankamai tiksliai.

Pateikiami keli potencialo pasiskirstymo bandinyje matematiniai modeliai ir atitinkami skaičiavimo algoritmai: pradedant pa-

prasčiausiu, kai pakanka atlikti kelis aritmetinius veiksmus, iki didelio tikslumo, kuriems realizuoti būtina šiuolaikinė skaičiavimo technika.

Visais atvejais išsamiai analizuojamos gaunamų paklaidų vertės ir jų struktūra, nurodomi matavimai, reikalingi atliekant fizikinius bandymus.